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# $L^1$-Convergence of Fourier Series with $O$-Regularly Varying Quasimonotonic Coefficients

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Among various generalizations of sequential monotonicity, the quasimonotonic sequences of O. Szasz [1] play a prominent role in the theory of $L^1$-convergence of Fourier series. As S. A. Telyakovskii and G. A. Fomin [2] have shown, quasimonotonic Fourier coefficients form an important $L^1$-convergence class.

A null sequence $\{a_n\}$ of real numbers is quasimonotonic if, for some $\alpha \geqslant 0$, the sequence $\{a_n/n^\alpha\}$ is monotonic. For $\alpha = 0$, quasimonotonicity reduces to monotonicity.

The significance of quasimonotonic sequences is well illustrated by the following result in [2].

THEOREM A. *Let*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1}$$

*be the Fourier series of some* $f \in L^1(0, \pi)$. *If* $\{a_n\}$ *is quasimonotonic then*

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty, \tag{2}$$

*is equivalent to*

$$a_n \lg n = o(1), \qquad n \to \infty, \tag{3}$$

*where* $S_n(f, x) = S_n(f) = a_0/2 + \sum_{k=1}^{n} a_k \cos kx$ *and* $\|\cdot\|$ *denotes* $L^1(0, \pi)$-*norm.*

A similar result is valid for the sine series.



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Recently an attempt has been made in [3] to extend the idea of quasimonotonic sequences and, consequently, to obtain a generalization of Theorem A. Instead of the sequence $\{n^\alpha\}$, $\alpha \geqslant 0$, W. O. Bray and Č. V. Stanojević [3] used a regularly varying sequence $\{\gamma(n)\}$ as a gauge. A non-decreasing sequence $\{\gamma(n)\}$ of positive numbers is regularly varying in the sense of J. Karamata [4] if for some $\rho \geqslant 0$

$$\lim_n \frac{\gamma([\lambda n])}{\gamma(n)} = \lambda^\rho, \qquad \lambda > 1.$$

A null sequence $\{a_n\}$ of real numbers is defined in [3] to be regularly varying quasimonotonic if for some regularly varying sequence $\{\gamma(n)\}$ the sequence $\{a_n/\gamma(n)\}$ is monotonic.

In [3] the following generalization of Theorem A is proposed.

THEOREM B. *Let* (1) *be the Fourier series of some* $f \in L^1(0, \pi)$. *If* $\{a_n\}$ *is a regularly varying quasimonotonic sequence, then* (2) *is equivalent to* (3).

An effort is made in [3] to prove Theorem B by estimating $\|S_n(f) - \sigma_n(f)\|$, where $\sigma_n(f)$ are $(c, 1)$-means of the partial sums $S_n(f)$ of (1). Apparently this is too coarse a method for such a substantial generalization of Theorem A. (At the end of the proof of sufficiency in [3] there is also a typographical error that makes the rest of the proof invalid.)

In this paper we shall show that Theorem B may be further generalized if sharper estimations are used. For this purpose we need a non-decreasing $O$-regularly varying sequence of positive numbers. A non-decreasing sequence $\{R(n)\}$ of positive numbers is $O$-regularly varying if

$$\overline{\lim_n} \frac{R([\lambda n])}{R(n)} \text{ is finite}, \qquad \text{for} \quad \lambda > 1.$$

A null sequence $\{a_n\}$ of real numbers is $O$-regularly varying quasimonotonic if for some non-decreasing $O$-regularly varying sequence $\{R(n)\}$ of positive numbers, the sequence $\{a_n/R(n)\}$ is monotonic. (Notice that the sequence $\{a_n\}$ is necessarily positive.)

We now have the following theorem.

THEOREM. *Let* $\{a_n\}$ *be an O-regularly varying quasimonotonic sequence, and for some even* $f \in L^1(0, \pi)$ *let* $\hat{f}(n) = a_n$ *for* $n = 0, 1, 2, \cdots$. *Then the necessary and sufficient condition for* $\|S_n(f) - f\| = o(1)$, $n \to \infty$, *is* $a_n \lg n = o(1)$, $n \to \infty$.

*Proof. Sufficiency.* Assume that $a_n \lg n = o(1)$, $n \to \infty$. For $\lambda > 1$ consider the identity

$$f(x) - S_n(f, x) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} \Delta a_j D_j(x) - \frac{[\lambda n]+1}{[\lambda n]-n} (\sigma_{[\lambda n]} - f) + \frac{n+1}{[\lambda n]-n} (\sigma_n - f) + \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} a_{k+1} D_k(x) - a_{n-1} D_n(x), \qquad (4)$$

where $D_n$ is the Dirichlet kernel. Since

$$\|D_n\| = \frac{4}{\pi^2} \lg n + O(1), \qquad n \to \infty,$$

then after taking the norm of both sides of (4), majorizing the right-hand side, and taking $n$ sufficiently large, we get

$$\|f - S_n(f)\| \leqslant \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} \sum_{j=n+1}^{k} |\Delta a_j| \left( \frac{4}{\pi^2} \lg j + O(1) \right) + \frac{[\lambda n]+1}{[\lambda n]-n} \|\sigma_{[\lambda n]}(f) - f\| + \frac{n+1}{[\lambda n]-n} \|\sigma_n(f) - f\| + \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} a_{k+1} \left( \frac{4}{\pi^2} \lg k + O(1) \right) + a_{n+1} \left( \frac{4}{\pi^2} \lg n + O(1) \right).$$

Majorizing the right-hand side of the last inequality we have

$$\|f - S_n(f)\| \leqslant 2 \sum_{k=n+1}^{[\lambda n]} |\Delta a_k| \lg k + \frac{[\lambda n]+1}{[\lambda n]-n} \|\sigma_{[\lambda n]}(f) - f\| + \frac{n+1}{[\lambda n]-n} \|\sigma_n(f) - f\| + \frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} a_{k+1} \lg(k+1) + o(1) + a_{n+1} \lg(n+1). \qquad (5)$$

Taking the limit superior of both sides of inequality (5) we obtain

$$\begin{aligned}\overline{\lim_n} \|f-S_n(f)\| \leqslant 2\,\overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} |\Delta a_k| \lg k &+ \overline{\lim_n} \frac{[\lambda n]+1}{[\lambda n]-n} \cdot \overline{\lim_n} \|\sigma_{[\lambda n]}(f)-f\| \\ &+ \overline{\lim_n} \frac{n+1}{[\lambda n]-n} \cdot \overline{\lim_n} \|\sigma_n(f)-f\| \\ &+ \overline{\lim_n} \left[\frac{1}{[\lambda n]-n} \sum_{k=n-1}^{[\lambda n]} a_{k+1} \lg(k+1)\right] \\ &+ \overline{\lim_n}\, a_{n+1} \lg(n+1).\end{aligned}$$

For $f \in L^1(0, \pi)$, the second and the third term in the above inequality are $o(1)$ as $n \to \infty$. The forth and the last term in the above inequality are also $o(1)$ as $n \to \infty$, according to our assumption $a_n \lg n = o(1)$, $n \to \infty$. Therefore it remains to estimate the first term in the above inequality and to show that it is also $o(1)$ as $n \to \infty$, for $O$-regularly varying quasimonotonic coefficients for which $a_n \lg n = o(1)$, $n \to \infty$.

From the monotonicity of the sequence $\{a_n/R(n)\}$ we get

$$\begin{aligned}\sum_{k=n+1}^{[\lambda n]} |\Delta a_k| \lg k &\leqslant R([\lambda n]) \lg[\lambda n] \sum_{k=n-1}^{[\lambda n]} \left(\Delta \frac{a_k}{R(k)}\right) \\ &+ \sum_{k=n-1}^{[\lambda n]} R(k) \left(\frac{1}{R(k)} - \frac{1}{R(k+1)}\right) a_{k+1} \lg(k+1) \\ &\leqslant R([\lambda n]) \lg[\lambda n] \left(\frac{a_{n+1}}{R(n+1)} - \frac{a_{[\lambda n]+1}}{R([\lambda n]+1)}\right) \\ &+ \max_{n+1 \leqslant k \leqslant [\lambda n]} (a_{k+1} \lg(k+1)) \sum_{k=n-1}^{[\lambda n]} \left(1 - \frac{R(k)}{R(k+1)}\right) \\ &\leqslant \frac{R([\lambda n])}{R(n)} \cdot \frac{\lg[\lambda n]}{\lg n} \cdot a_{n+1} \lg(n+1) \\ &+ \max_{n+1 \leqslant k \leqslant [\lambda n]} (a_{k+1} \lg(k+1)) \\ &\times \frac{R([\lambda n])}{R(n)} \sum_{k=n+1}^{[\lambda n]} \left(\frac{R(k+1)}{R(k)} - 1\right).\end{aligned}$$

Taking the limit superior of both sides of the last inequality we have

$$\overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} |\Delta a_k| \lg k \leqslant \overline{\lim_n} \frac{R([\lambda n])}{R(n)} \cdot \overline{\lim_n} \frac{\lg[\lambda n]}{\lg n} \cdot \overline{\lim_n} (a_{n+1} \lg(n+1))$$

$$+ \overline{\lim_n} \max_{n+1 \leqslant k \leqslant [\lambda n]} (a_{k+1} \lg(k+1)) \cdot \overline{\lim_n} \frac{R([\lambda n])}{R(n)}$$

$$\cdot \overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{R(k+1)}{R(k)} - 1 \right).$$

Recalling that $\overline{\lim}_n (R([\lambda n])/R(n))$ is finite for $\lambda > 1$, and $\lim_n (\lg[\lambda n]/\lg n) = 1$, and that $a_n \lg n = o(1)$, $n \to \infty$, holds, we conclude that the first term on the right-hand side of the last inequality is $o(1)$, as $n \to \infty$. It remains to show that

$$\overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{R(k+1)}{R(k)} - 1 \right) \text{ is finite,} \qquad \text{for} \quad \lambda > 1. \tag{6}$$

Indeed, from

$$\sum_{k=n+1}^{[\lambda n]} \left( \frac{R(k)}{R(k-1)} - 1 \right) \leqslant \prod_{k=n+1}^{[\lambda n]} \left[ 1 + \left( \frac{R(k)}{R(k-1)} - 1 \right) \right] = \frac{R([\lambda n])}{R(n)}$$

it follows that

$$\overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} \left( \frac{R(k)}{R(k-1)} - 1 \right) \leqslant \overline{\lim_n} \frac{R([\lambda n])}{R(n)}.$$

(As a matter of fact the condition (6) is a necessary and sufficient condition for a non-decreasing sequence $\{R(n)\}$ of positive numbers to be an $O$-regularly varying sequence.) This completes the proof of sufficiency.

*Necessity.* Suppose that $\|S_n(f) - f\| = o(1)$, $n \to \infty$, holds. From the well-known inequality

$$\|S_n(f) - f\| \geqslant \sum_{k=1}^{n} \frac{a_{n+k}}{k},$$

and from the fact that $(a_n/R(n))\downarrow$, we get

$$\|S_n(f) - f\| \geqslant \frac{a_{2n}}{R(2n)} \sum_{k=1}^{n} \frac{R(n+k)}{k} \geqslant C\, a_{2n} \lg n \frac{R(n)}{R(2n)},$$

where $C$ is an absolute constant. Thus

$$C\, a_{2n} \lg n \leqslant \frac{R(2n)}{R(n)} \|S_n(f) - f\|.$$

Taking the limit superior of both sides of the last inequality, the proof of necessity follows.

Clearly, Theorem B is a corollary to our Theorem.

It would be of interest to prove a theorem corresponding to the above result in the case of Banach space $L^1(T)$, $T = \mathbb{R}/2\pi\mathbb{Z}$, of complex valued Lebesque integrable functions, following the ideas of complex sequential monotonicity [5, 6].

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